All-Shortest-Path 2-Interval Routing is NP-Complete

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Abstract

k-Interval Routing Scheme (k-IRS) is a compact routing method that allows up to k interval labels to be assigned to an arc. A fundamental problem is to characterize the networks that admit k-IRS. All of the problems related to single-shortest-path k-IRS have already been shown to be NP-complete. For all-shortest-path k-IRS, the characterization problems have been proved to be NP-complete for every $k \geqslant 3$, and remain open for $k = 1, 2$. In this paper, we close the open case of $k = 2$ by showing that it is NP-complete to decide whether a graph admits an all-shortest-path 2-IRS. The same proof is also valid for all-shortest-path Strict 2-IRS. All-shortest-path Strict k-IRS is previously known to be polynomial for $k = 1$, open for $k = 2, 3$, and NP-complete for every constant $k \geqslant 4$.

Key words: Interval routing schemes; compact routing; NP-completeness.

1 Introduction

Interval Routing is a space-efficient routing method for communication networks [15]. The routing table stored at each node groups the set of destination addresses that use the same output port into intervals of consecutive addresses. Formally, the network is modeled as a

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finite graph $G = (V, E)$, where the set of vertices, V, represents the nodes of the network, and the set of edges, E, represents the bidirectional links. An edge $\{u, v\}$ between the nodes u and v induces two opposite arcs: (u, v) and (v, u); the $2|E|$ induced arcs form an arc set A of G. A routing scheme for network G assigns each arc (u, v) a subset $I(u, v) \subseteq V$, such that the union of the subsets assigned to the arcs emanating from u covers the set $V - \{u\}$. Routing is then performed according to the assignment I, such that at vertex u, a message will be sent on the arc (u, v) whose $I(u, v)$ contains the destination of the message. A good interval routing scheme would try to minimize the number of intervals in $I(u, v)$ over all the arcs by selecting a particular address mapping $L : V \to \{1, 2, \ldots, |V|\}$ and an assignment $I : A \to 2^V$. If each $I(u, v)$ contains no more than k intervals under L, the routing scheme, denoted by $R = \langle L, I \rangle$, is called a *k-Interval Routing Scheme* (k-IRS).

The standard definition of IRS assumes a single routing path between any two nodes, which imposes the outgoing arcs of a node u to be assigned disjoint subsets, i.e., $I(u, v) \cap I(u, w) = \phi$ for $v \neq w$. Clearly, the routing process with such an IRS is deterministic. A more flexible routing scheme, called *multipath* IRS or *non-deterministic* IRS [16, 10], allows multiple arcs of a node to lead to the same destination; the routing process can pick one of these arcs arbitrarily or according to traffic conditions.

In general, routing along shortest paths is desirable. A *shortest-path* IRS always induces shortest paths. A *single-shortest-path* IRS offers a *unique* shortest path between any two vertices in the graph. An *all-shortest-path* IRS is a multipath IRS that gives exactly *all* the shortest paths between any pair of vertices in the graph.

1 and $|V|$ being considered consecutive, the interval $[a, b]$ with $a > b$ denotes the set $\{i \mid a \leqslant i \leqslant |V|\} \cup \{i \mid 1 \leqslant i \leqslant b\}$. An IRS is *strict*, denoted by SIRS, if every arc (u, v) satisfies $u \notin I(u, v)$.

The space efficiency of an IRS is measured by *compactness*. Compactness is the maximum, over all the arcs (u, v), of the number of intervals in $I(u, v)$. The characterization of networks that admit a *shortest-path* interval routing scheme with compactness k (i.e., k-IRS) is a fundamental question in this field[1]. Successful work has been done for many special classes of graphs, including trees, outerplannar graphs, hypercubes, meshes, r-partite graphs, interval graphs, unit-circular graphs, tori, 2-trees, chordal rings, and general graphs; see [1, 8, 13, 14, 15, 16, 17] for some examples. A summary of these and other results can be found in [10, 11]. For general graphs, existing complexity results for IRS and SIRS are summarized in Table 1, where NPC denotes an NP-complete problem.

All the problems related to single-shortest-path IRS are known to be NP-complete. The NP-completeness for single-path with fixed $k \geqslant 3$ follows from a combination of [6] with [12], or with [18]. In [6], Flammini gave a polynomial-time construction of graphs from binary matrices such that there are at most k blocks of consecutive 1's in each column of the matrix under some row permutation if and only if there is a single-shortest-path $(k+1)$-IRS for the constructed graph; in [12], Goldberg et al. proved that for every constant $k \geqslant 2$, deciding whether a given binary matrix can be row permuted such that each column has at most k

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[1] Note the stress on "shortest-path" because if the shortest path requirement is relaxed, every graph supports a single-path 1-IRS.

Table 1: Complexity results on characterization of shortest-path IRSes.

Paths represented	Compactness k	Variants	Results
Single	fixed $k = 1$	IRS, SIRS	NPC, [3]
	fixed $k = 2$	IRS, SIRS	NPC, [6]
	fixed $k \geqslant 3$	IRS, SIRS	NPC, [6, 12, 18]
	general k	IRS, SIRS	NPC, [6, 3]
All	fixed $k = 1$	IRS	open
		SIRS	P, [2, 7]
	fixed $k = 2$	IRS, SIRS	NPC, this paper
	fixed $k = 3$	IRS	NPC, [19]
		SIRS	open
	fixed $k \geqslant 4$	IRS, SIRS	NPC, [19]
	general k	IRS, SIRS	NPC, [19, 5]

blocks of consecutive 1's is NP-complete; in [18], Wang and Lau strengthened the result by showing that the same NP-completeness holds even if the problem is restricted to symmetric matrices.

For the all-shortest-path IRS case, only partial answers (both positive and negative) have been given. 1-SIRS can be reduced to the consecutive ones property of binary matrices, which can be solved in linear time [2]. For the case that 1 and n is considered nonconsecutive, Flammini et al. in [7] presented efficient characterizations for 1-IRS and 1-SIRS (a.k.a. 1-LIRS and 1-SLIRS). On the negative side, in [5], it was shown that the optimization problem of determining the minimal k such that a given weighted network belongs to the class of all-shortest-path k-IRS is NP-hard. Wang et al. in [19] got the same result and proved that the all-shortest-path k-IRS problem is NP-complete for every constant $k \geqslant 3$ and the all-shortest-path k-SIRS problem is NP-complete for every constant $k \geqslant 4$, even the networks are restricted to be unweighted. The characterization remains open for all-shortest-path IRS of compactness $k = 1, 2$ and for all-shortest-path SIRS of compactness $k = 2, 3$.

In this paper, we prove that the characterization of networks which admit all-shortest-path k-IRS (k-SIRS) is NP-complete for $k = 2$.

The rest of the paper is organized as follows. The next section gives some formal definitions of the IRS models. The NP-completeness results are presented in Section 3. We give some conclusive remarks in the last section.

2 Preliminaries

The graphs we consider are connected, loopless, and do not contain multi-edges. We use $Adj(v)$ to denote the set of neighbors of vertex v in the graph.

We define an Interval Routing Scheme (IRS) as follows.

Definition 1 *Let $G = (V, E)$ be a graph and A is the arc set induced from E. An IRS on G is a pair $\langle L, I \rangle$ where*

(1) *L is a one-to-one vertex labeling, $L : V \to \{1, 2, \ldots, |V|\}$;*

(2) *I is an arc labeling, $I : A \to 2^V$, assigning a subset of V to each arc of A, such that for every vertex $u \in V$, $\bigcup_{(u,v) \in A} I(u, v) \bigcup \{u\} = V$;*

(3) *for every $x, y \in V$:*

  (3.1) *there exists a sequence of vertices $x = u_0, u_1, \ldots, u_s = y$ such that for $1 \leqslant i \leqslant s$, $(u_{i-1}, u_i) \in A$ and $y \in I(u_{i-1}, u_i)$; this sequence is called a routing path induced by $\langle L, I \rangle$;*

  (3.2) *any routing path induced by $\langle L, I \rangle$ between x and y is a simple path of G, i.e., $u_0, u_1, \ldots, u_s$ are mutually different vertices of V.*

To save space in the routing table, an IRS expresses $I(u, v)$, the subset of V assigned to an arc $e = (u, v)$, with intervals over $\{1, 2, \ldots, |V|\}$.

Definition 2 *An interval of $\{1, 2, \ldots, |V|\}$ is one of the following:*

(1) *A linear interval $[i, j] = \{i, i + 1, \ldots, j\}$, where $i, j \in \{1, 2, \ldots, |V|\}$ and $i \leqslant j$;*

(2) *a circular interval $[i, j] = \{i, \ldots, |V|, 1, \ldots, j\}$, where $i, j \in \{1, \ldots, |V|\}$ and $i > j$; or*

(3) *the null interval $[\ ]$ which is the empty set ϕ.*

For simplicity, we will not always strictly distinguish between a vertex v and its label $L(v)$, and will say that a vertex $v \in V$ is contained in an interval $[i, j]$ if $L(v) \in [i, j]$.

Definition 3 *Given $U \subseteq V$ and a labeling L of V, we denote by $N(L, U)$ the minimum number of disjoint intervals such that their union is equal to $\{L(v) \bigm| v \in U\}$.*

For example, suppose $V = \{v_1, v_2, \ldots, v_9\}$ and $L(v_i) = i$, then we have

$$N(L, \{v_1, v_2, v_5, v_6, v_7, v_9\}) = 2$$

because $\{1, 2, 5, 6, 7, 9\} = [9, 2] \cup [5, 7]$.

IRS allows $I(u, v)$ to include the starting vertex u. If $u \in I(u, v)$, there is an interval on arc (u, v) that contains the starting vertex u. An interval on an outgoing arc (u, v) of a vertex u containing the vertex u is called an *self-enclosing* interval, otherwise it is a *strict* interval. A variant of IRS is called Strict Interval Routing Scheme ($SIRS$), which restricts to use strict intervals only, i.e., for every arc $(u, v) \in A$, $u \notin I(u, v)$.

Definition 4 *Let $R = (L, I)$ be an IRS (SIRS) on a graph $G = (V, E)$, then the compactness of R is the integer $k = \max\{N(L, I(u, v)), N(L, I(v, u)) \bigm| u, v \in E\}$. We denote by k-IRS (k-SIRS) every IRS (SIRS) of compactness no more than k.*

For practical reasons, we are interested in designing IRSes that induce only shortest paths. For an arc $e = (u, v)$, we let $S(u, v)$ denote the subset of vertices which can be reached from vertex u through a shortest path starting with e; note that $S(u, v) \neq S(v, u)$.

Definition 5 *Let $R = \langle L, I \rangle$ be an IRS (SIRS) on a graph $G = (V, E)$; we say that R is*

(1) *a single-shortest-path IRS (SIRS) if it induces one and only one of the shortest paths between every pair $x, y \in V$; or*

(2) *an all-shortest-path IRS (SIRS) if it induces exactly the set of all shortest paths between every pair $x, y \in V$.*

By the definitions, for all-shortest-path IRS, the arc labeling I does not have much flexibility in assigning subsets to arcs—$I(u, v)$ is either $S(u, v)$ or $S(u, v) \cup \{u\}$; and for all-shortest-path SIRS, the arc labeling I is identical to S, i.e., $I(u, v) = S(u, v)$ for every arc (u, v).

Given a graph G and an integer k, the problem of determining whether G supports an all-shortest-path k-IRS (k-SIRS) is named the k-IRS (k-SIRS) problem. This problem is in the class of NP. Given a graph G, an integer k, a vertex labeling L, and an arc labeling I, it can be verified in polynomial time whether $\langle L, I \rangle$ is an all-shortest-path k-IRS (k-SIRS) for G.

3 The NP-completeness Results

This section is to prove the NP-completeness for 2-IRS and 2-SIRS. The proof is based on the following NP-complete problem [9].

Hamiltonian Circuit in Cubic Graphs (HCCG)

Instance: A cubic graph $G = (V, E)$.[2]

Question: Does G contains a Hamiltonian circuit?

In fact, the result given in [9] is much stronger: they proved that the well-known Hamiltonian Circuit Problem remains NP-complete when restricted to graphs that are cubic, 3-connected, planar, and have no face with fewer than five edges. We first in Subsection 3.1 reduce HCCG to the following HPCG problem.

Hamiltonian Path in Cubic Graphs (HPCG)

Instance: The girth of a graph is the length of a shortest cycle contained in the graph. A graph $G = (V, E)$ with girth $g(G) \geqslant 5$ and that is cubic except two specified vertices a and b which are of degree 2.

Instance: Does G contain a Hamiltonian path between a and b?

We have already proved HPCG to be NP-complete in [18, 20]. For self-contained, we introduce the proof briefly in Subsection 3.1. In Subsection 3.2, we give some useful facts about the instance graph of HPCG. After that, we transform HPCG to 2-IRS and 2-SIRS in Subsection 3.3. Fig. 1 shows the polynomial transformations among these problems. The numbers with the arrows are the subsections where we present the transformations.

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[2] *cubic* is synonymous with *3-regular*, i.e., every vertex is of degree 3.

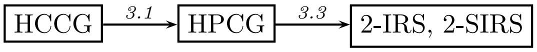


Figure 1: Transformations among the problems.

3.1 From HCCG to HPCG

Starting from any cubic graph G, an instance of HCCG, we present a way to construct in polynomial time an instance of HPCG–a graph G^* that is cubic but for two degree-2 vertices a and b, and has a girth $g(G^*) \geqslant 5$, such that G has Hamiltonian circuit if and only if G^* has a Hamiltonian path from a to b. The construction proceeds in two steps.

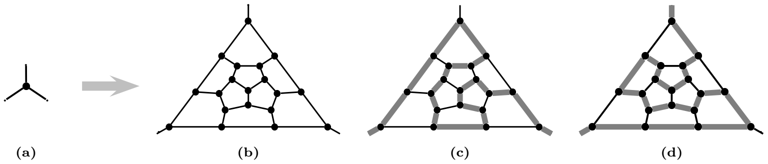


Figure 2: The replaced vertex (a) and the substitution graph (b) with possible local paths (c) and (d) (alternate and symmetric paths omitted). A Hamiltonian circuit, once entering the substitution graph, must visit all 19 vertices before leaving the substitution graph.

The first step is to make the girth of the graph to be at least 5 while keeping the cubic properties and the Hamiltonicity. This is achieved by replacing each vertex of G with the substitution graph shown in Fig. 2 (due to [9]). The resulting graph G' obviously remains cubic and has a girth $g(G') \geqslant 5$. Moreover, G' and G concurrently have or have no Hamiltonian circuits.

The second step is to choose an arbitrary vertex v in G' to be replaced by the substitution graph shown in Fig. 3. Obviously, this will result in a graph G^* that remains girth $g(G^*) \geqslant 5$. G^* is cubic except for the two special vertices a and b which are of degree 2. G^* is an instance of HPCG. Moreover, because every Hamiltonian circuit (if any) in G^* must pass through the edge $\{a, b\}$, G^* has a Hamiltonian path from a to b if and only if G' has a Hamiltonian circuit, and if and only if G has a Hamiltonian circuit.

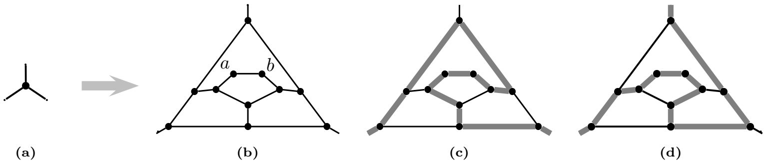


Figure 3: The replaced vertex (a) and the substitution graph (b) with possible local paths (c) and (d) (alternate and symmetric paths omitted). A Hamiltonian circuit, once entering the substitution graph, must pass through the edge $\{a, b\}$ and visit all 11 vertices before leaving the substitution graph.

It is easy to see that the construction can complete within polynomial time, thus we can close this subsection with the next theorem.

Theorem 1 *HPCG is NP-complete.*

3.2 Some properties of the instance graph of HPCG

The proof of NP-completeness of 2-IRS and 2-SIRS will use some special properties of the instance graph G of HPCG. To state these properties, we first need to define some terminologies.

Given a graph $G = (V, E)$ and an ordering $P = v_1v_2 \ldots v_n$ that orders all the vertices in G, we term (v_{i-1}, v_i) $(1 < i \leqslant n)$ a *consecutive* pair (on P). A consecutive pair (v_{i-1}, v_i) is a *gap* (on P) if edge $\{v_{i-1}, v_i\} \notin E$. In particular, though we do not view the pair (v_1, v_n) as consecutive, we always view (v_1, v_n) as a gap of P, no matter $\{v_1, v_n\} \in E$ or not. The former type of gaps are called *inner* gaps, and the later *outer* gap. Note that a gap is not necessarily a consecutive pair but an inner gap must be. Obviously, a gap (u, v) involves exactly two vertices u and v, a vertex involves at most two gaps, the number of inner gaps can vary from 0 to $n - 1$, while that of the outer ones is a constant 1. For any pair (v_i, v_j), if there is a vertex v_k such that both $\{v_k, v_i\}$ and $\{v_k, v_j\}$ are edges of G, we term that v_k covers the pair (v_i, v_j).

If $G = (V, E)$ is a graph with girth $g(G) \geqslant 5$, and $P = v_1v_2 \ldots v_n$ a vertex ordering of G, then we can easily note the following facts.

Fact 1: G has more than 3 vertices, i.e., $n > 3$.

Fact 2: If (v_{i-1}, v_i) is not a gap, then no vertex can covers (v_{i-1}, v_i) (otherwise if v_j covers it, then G has a circuit $v_jv_{i-1}v_iv_j$ of length 3, contradicting to $g(G) \geqslant 5$.

Fact 3: A gap (v_{i-1}, v_i) can not be covered by more than one vertex. (otherwise, if both v_j and v_k cover (v_{i-1}, v_i), then there will be in G a circuit $v_jv_{i-1}v_kv_iv_j$ of length 4, contradicting to $g(G) \geqslant 5$).

For the instance graph G of HPCG, we have the next lemma to state a sufficient condition for the problem to answer "yes".

Lemma 1 *Let $P = v_1v_2 \ldots v_n$ be a vertex ordering of a graph $G = (V, E)$ that is cubic but for two degree-2 vertices and have a girth $g(G) \geqslant 5$, if for any degree-3 vertex the number of the inner gaps it covers is at least the number of the gaps it involves in P, then there must exist in G a Hamiltonian path between the two degree-2 vertices.*

Proof: Suppose that there are X inner gaps, i.e., totally $X + 1$ gaps on P. Let us denote by $X_{i,j}$ the number of degree-i vertices each of which involves in exactly j gaps (inner or outer). Then each vertex in $X_{3,2}$ must cover at least two inner gaps, and each of $X_{3,1}$ at least one. By Fact 3 two vertices cannot cover the same gap, there should be at least $2X_{3,2} + X_{3,1}$ inner gaps for the degree-3 gap-involved vertices to cover. So

$$X \geqslant 2X_{3,2} + X_{3,1}. \tag{1}$$

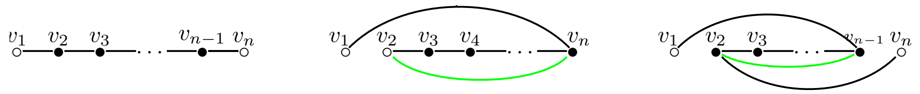
Note that $2\,(X_{2,2} + X_{3,2}) + X_{2,1} + X_{3,1}$ is the number that counts each gap twice. So

$$2\,(X_{2,2} + X_{3,2}) + X_{2,1} + X_{3,1} = 2(X + 1). \tag{2}$$

The above implies $X_{2,1} + X_{3,1}$ is even. If $X_{2,1} + X_{3,1} = 0$ then there would be n gaps, and each of the $n-2$ degree-3 vertices would involve in two gaps, and thus there should be at least $2(n-2)$ inner gaps for the degree-3 vertices to cover, leading to $2(n-2) \leqslant n-1$, i.e., $n \leqslant 3$. contradicting to Fact 1. So, it must be that

$$X_{2,1} + X_{3,1} \geqslant 2. \tag{3}$$

Obviously $X_{2,2} + X_{2,1} \leqslant 2$ because there are only two degree-2 vertices. Combining this with (1)–(3) gives ($X_{3,2}$, $X_{2,2}$, $X_{3,1}$, $X_{2,1}$, X) only three non-negative integer solutions: (1) (0, 0, 0, 2, 0), (2) (0, 1, 1, 1, 1), and (3) (0, 2, 2, 0, 2). There is no degree-3 vertex involving in more than one gap. Considering that the degree-3 vertex involving a gap must cover an inner gap it doesn't involves in, the possible cases (symmetric cases omitted) for each solution are all depicted in Fig. 4. In the figure the circled nodes represent vertices of degree 2 and the solid ones degree 3. It can be seen that for each case there must be a Hamiltonian path between the two circled nodes, the two degree-2 vertices. □



Solution 1
(0, 0, 0, 2, 0)
HP: $v_1 v_2 \ldots v_n$

Solution 2
(0, 1, 1, 1, 1)
HP: $v_1 v_n v_{n-1} \ldots v_3 v_2$

Solution 3
(0, 2, 2, 0, 2)
HP: $v_1 v_{n-1} v_{n-2} \ldots v_3 v_2 v_n$

Figure 4: The three solutions for $(X_{3,2}, X_{2,2}, X_{3,1}, X_{2,1}, X)$. ○–vertex of degree 2, ●–vertex of degree 3.

3.3 From HPCG to 2-IRS and 2-SIRS

The transformation from HPCG to 2-IRS and that from HPCG to 2-SIRS is the same one. Starting with any instance of HPCG, $G = (V, E)$, a cubic but for two degree-2 vertices a and b graph with $g(G) \geqslant 5$, the instance graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ of 2-IRS (as well as of 2-SIRS) is simply obtained by adding one new vertex x to G to link all the other vertices. Formally,

$$\begin{aligned} \mathbb{V} &= V \cup \{x\}, \\ \mathbb{E} &= E \cup \big\{\{x, v\} \bigm| v \in V\big\}. \end{aligned}$$

Lemma 2 *For a graph $G = (V, E)$ that has $g(G) \geqslant 5$ and that is cubic except two degree-2 vertices a and b, G has a Hamiltonian path from a to b if and only if the graph $\mathbb{G} = (\mathbb{V}, \mathbb{E})$ obtained by the transformation from G supports an all-shortest-path 2-IRS.*

Proof: Note that the added vertex x limits the diameter of $\mathbb{G}$ to be 2, so that for each arc $e = (u, v)$, $S(e)$, the optimally reachable vertices from u via e, is $Adj_{\mathbb{G}}(v) \backslash Adj_{\mathbb{G}}(u) \cup \{v\} \backslash \{u\}$. Thus, the $S(e)$ for various types of arcs e in $\mathbb{G}$ can be summarized in the following.

on (x, v_i)**:** $S(x, v_i) = \{v_i\}$, which, independently with any vertex labeling, always receives a single interval;

on (v_i, x)**:** $S(v_i, x) = V \setminus \mathit{Adj}_G(v_i) \cup \{x\} \setminus \{v_i\} = \mathbb{V} \setminus (\mathit{Adj}_G(v_i) \cup \{v_i\})$;

on (v_i, v_j)**:** $S(v_i, v_j) = (\mathit{Adj}_G(v_j) \cup \{v_j\}) \setminus \{v_i\}$. This holds because from $g(G) \geqslant 5$ we can infer that $\mathit{Adj}_G(v_i) \cap \mathit{Adj}_G(v_j) = \phi$ if $(v_i, v_j) \in A$ (otherwise there would be triangles in G).

Let us first prove the *only if* part of the lemma. Suppose in G there exists a Hamiltonian path $P = v_1 v_2 \dots v_n$ where $v_1 = a$ and $v_n = b$. Let us consider a labeling L that labels $\mathbb{V}$ in the order $v_1 v_2 \dots v_n x$, and the assignment I such that $I(e) = S(e)$ (note that this is an SIRS). Then,

- for the arc (x, v_i), independently with any vertex labeling, it always receives a single interval;
- for the arc (v_i, x), because

  $$N\left(L, \mathbb{V} \setminus (\mathit{Adj}_G(v_i) \cup \{v_i\})\right) = N\left(L, \mathit{Adj}_G(v_i) \cup \{v_i\}\right),$$

  it receives the same number of intervals as that for $\mathit{Adj}_G(v_i) \cup \{v_i\}$; but $\mathit{Adj}_G(v_i) \cup \{v_i\} = \{v_{i-1}, v_i, v_{i+1}\} \cup \{v_k\}$ (or $\{v_i, v_{i+1}\} \cup \{v_k\}$ if $i = 1$ or $\{v_{i-1}, v_i\} \cup \{v_k\}$ if $i = n$, v_k is another vertex in G linked v_i), which under L forms at most 2 intervals, $[i-1, i+1]$ and $[k, k]$ (or $[i, i+1]$ and $[k, k]$ if $i = 1$, or $[i-1, i]$ and $[k, k]$ if $i = n$);
- for the arc (v_i, v_j), either $i = j - 1$ or $i = j + 1$ or $i = k$ (v_k is another vertex in G linked v_j). If $i = j - 1$ the intervals on the arc are $[j, j+1]$ (or $[j, j]$ if $j = n$) and $[k, k]$; if $i = j + 1$ the intervals are $[j-1, j]$ (or $[j, j]$ if $j = 1$) and $[k, k]$; otherwise only $[j-1, j+1]$ (or $[j, j+1]$ if $j = 1$, or $[j-1, j]$ if $j = n$). Every case shows that the arc $(v_i, v_j$ receives no more than two intervals.

So, $\langle L, I\rangle$ is 2-IRS for $\mathbb{G}$. The *only if* part of the lemma is proved.

To prove the *if* part of the lemma, suppose that $\langle L, I\rangle$ is an all-shortest-path *2-IRS* for $\mathbb{G}$, where L labels $\mathbb{V}$ in the order $v_{i+1} v_{i+2} \dots v_n x v_1 v_2 \dots v_i$. Then we will apply Lemma 1 on $P = v_1 v_2 \dots v_n$ which is an ordering of the vertices of G.

We claim that a degree-3 vertex v_i that involves in one gap must cover an inner gap. To justify this, suppose, without loss of generality, (v_{i-1}, v_i) is a gap and (v_i, v_{i+1}) not, and $\mathit{Adj}_G(v_i) = \{v_{i+1}, v_j, v_k\}$. If every two of $\{v_{i+1}, v_j, v_k\}$ is not adjacent on P, the arc (v_{i+1}, v_i) will have to receive 3 intervals $[i, i]$ (or $[i, i+1]$ if using self-enclosing interval), $[j, j]$, and $[k, k]$, contradicting to that $\langle L, I\rangle$ is a 2-IRS. Thus at least two of $\{v_{i+1}, v_j, v_k\}$ are adjacent on P. Since the two vertices adjacent on P link to v_i, by Fact 2, they form an inner gap covered by v_i.

We claim that a degree-3 vertex v_i that involves in two gaps must cover two inner gaps. To justify this, suppose $\mathit{Adj}_G(v_i) = \{v_j, v_k, v_l\}$ where $j < k < l$, then if either $k \neq j + 1$ or $k \neq l - 1$, then the arc (v_k, v_i) under L will receives three blocks $[i, i]$, $[j, j]$, and $[l, l]$ (self-enclosing interval in this case does not help), contradicting to that $\langle L, I\rangle$ is a 2-IRS. So, we must have $k = j + 1$ and $l = k + 1$. By Fact 2, both (v_j, v_k) and (v_k, v_l) are inner gaps covered by v_i.

Thus, Lemma 1 is applicable on $P = v_1 v_2 \dots v_n$ to guarantee us a Hamiltonian path between the two degree-2 vertices exists in G. $\Box$

By the NP-completeness of HPCG, the above lemma obviously implies the following theorem

Theorem 2 *All-shortest-path* 2*-IRS is NP-complete.*

Note that, in the *only if* part of the proof of Lemma 2, the 2-IRS for G derived from the Hamiltonian path is strict. Thus, the following holds.

Theorem 3 *All-shortest-path 2-SIRS is NP-complete.*

4 Discussion

We have proved that to recognize networks that admit all-shortest-path 2-IRS (2-SIRS) is NP-complete for unweighted graphs, and therefore also for weighted graphs. The result clearly implies that we cannot in polynomial time approximate the compactness of IRS (SIRS) within a ratio of less than 3/2, unless P=NP.

An IRS using only linear intervals is a *linear* IRS, LIRS for short. An IRS is denoted by SLIRS if it is both strict and linear. 1-LIRS and 1-SLIRS have been proved in *P* [7]. After the research in this paper, for all-shortest-path IRS and its variants, there are still four problems remain open. They are 1-IRS, 2-LIRS, 2-SLIRS, 3-SIRS, and 3-SLIRS.

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