Second-Order Equations
Section 6.3
Differential Equations [1]

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Spring 2009

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[1] These class-notes are based on the textbook “Differential Equations” by Paul Blanchard, Robert L. Devaney and Glen R. Hall

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1. Laplace transform of Sine and Cosine

In this section we study the non-autonomous differential equation

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = f(t) \tag{1}$$

Where p and q are constants and $f(t)$ is the forcing term. Recall that the case $f(t) = 0$ is well understood as a particular linear system. Also recall that we have seen nonlinear oscillations with periodic forcing that have chaotic behavior.

We have seen these two basic formulas for Laplace transforms.

$$\mathfrak{L}[sin(\omega\ t)] = \frac{\omega}{s^2 + \omega^2}$$

$$\mathfrak{L}[cos(\omega\ t)] = \frac{s}{(s^2 + \omega^2)}$$

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2. Shifting the Origin on the s-Axis

In this section we look at the Laplace transform:

$$\mathfrak{L}[e^{at}\ f(t)]$$

$$\mathfrak{L}[e^{at}\ f(t)] = F(s-a)$$

where $F(t)$ is the Laplace transform for $f(t)$.

3. A Forced Harmonic Oscillator

In chapter 4 the forced harmonic oscillator equation was introduced as a model of a mass attached to a spring sliding back and forth on a table with an external force caused by, for example, tilting the table.

Example:

As an example of the Laplace transform method for such equation, consider the initial-value

$$\begin{aligned} &\frac{d^2y}{dt^2} + 2y = cos(\omega t), \\ &y(0) = 0,\ y'(0) = 0 \qquad\qquad (\omega^2 \neq 2) \end{aligned} \tag{2}$$

The unforced harmonic oscillator

$$\frac{d^2y}{dt^2} + 2y = 0$$

(with natural period $T = \frac{2\pi}{\sqrt{2}}$) and forcing $cos(\omega t)$ with period $\frac{2\pi}{\omega}$.

To apply the Laplace method on the initial-value problem (2) we take the Laplace transform of both sides of the differential equation in (2), obtaining

$$\mathfrak{L}\left[\frac{d^2y}{dt^2}\right] + 2\mathfrak{L}[y] = \mathfrak{L}[cos(\omega t)].$$

Simplifying, we have

$$s^2\mathfrak{L}[y] - s\,\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0} + 2\mathfrak{L}[y] = \frac{s}{s^2+\omega^2}.$$

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Hence, we have

$$\begin{aligned}(s^2+2)\mathfrak{L}[y] &= \frac{s}{s^2+\omega^2}\\ \mathfrak{L}[y] &= \frac{s}{(s^2+2)(s^2+\omega^2)}\end{aligned}$$

If $\omega^2 \neq 2$ we will decompose using partial fractions

$$\frac{s}{(s^2+2)(s^2+\omega^2)} = \frac{As+B}{s^2+2} + \frac{Cs+D}{s^2+\omega^2}$$

Adding the fractions on the right hand side and then eliminating the denominators we have

$$s = \big(As+B\big)(s^2+\omega^2) + \big(Cs+D\big)(s^2+2)$$

Next expanding the terms on the left side obtain the system

$$\begin{aligned} A+C &= 0 \qquad \Rightarrow C = -A,\\ B+D &= 0\\ B\omega^2+2D &= 0 \qquad \omega^2 \neq 2 \Rightarrow B = D = 0\\ A\omega^2+2C &= 1 \end{aligned}$$

where

$$A = \frac{1}{\omega^2-2}; \quad C = -\frac{1}{\omega^2-2}$$

So that,

$$\frac{s}{(s^2+2)(s^2+\omega^2)} = \frac{1}{\omega^2-2}\left(\frac{s}{s^2+2} - \frac{s}{s^2+\omega^2}\right)$$

thus,

$$y(t) = \mathfrak{L}^{-1}\left[\frac{s}{(s^2+2)(s^2+\omega^2)}\right] = \frac{1}{\omega^2-2}\mathfrak{L}^{-1}\left[\frac{s}{s^2+2}\right] - \mathfrak{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right]$$

$$y(t) = \frac{1}{\omega^2-2}\left(cos(\sqrt{2}\,t) - cos(\omega t)\right) \tag{3}$$

Next, the question is: What is happening when $\omega \to \sqrt{2}$ Figure 1 (the forcing period gets closer to the natural period).

4. The Resonance Phenomenon

Let's try to understand the system (3) when $\omega \to \sqrt{2}$.

Here is a graph of the function $y(t)$ with $\omega = 1.2$ (sufficiently close to $\sqrt{2}$) see Figure 5. The amplitude of the oscillation increases and decreases in a very regular pattern. This phenomenon is called "*beating*", (see page 411) (which occurs when the natural response and the forced response have approximately the same frequency (or period). The two responses tend to reinforce or cancel each other over long time intervals. Then we have the following question

What about if $\omega = \sqrt{2}$?

In this case we have

$$\mathfrak{L}[y] = \frac{s}{(s^2+2)^2}$$

$$\Downarrow$$

$$y(t) = \mathfrak{L}^{-1}\left[\frac{s}{(s^2+2)^2}\right]$$

We haven't seen yet the inverse of this function. But the following formula is known

$$\mathfrak{L}[tsin(\omega t)] = \frac{2\omega s}{(s^2+\omega^2)^2}.$$

Then

$$\mathfrak{L}^{-1}\left[\frac{s}{(s^2+2)^2}\right] = \frac{1}{2\sqrt{2}}\,\mathfrak{L}^{-1}\left[\frac{2\sqrt{2}\;s}{(s^2+2)^2}\right] = \frac{1}{2\sqrt{2}}\;t\;sin(\sqrt{2}\;t)$$

Hence,

$$y(t) = \frac{1}{2\sqrt{2}}\, t\; sin(\sqrt{2}\, t).$$

The amplitude of the oscillation grows linearly see Figure 6 (Figure 4.21 page 417 in the book) .

Forcing with a frequency (period) equal to the natural frequency is called *resonant forcing*. The oscillator is said to be in *resonance*, and leads to indefinitely large amplitude. For practical reasons this case has to be avoided at all cost!

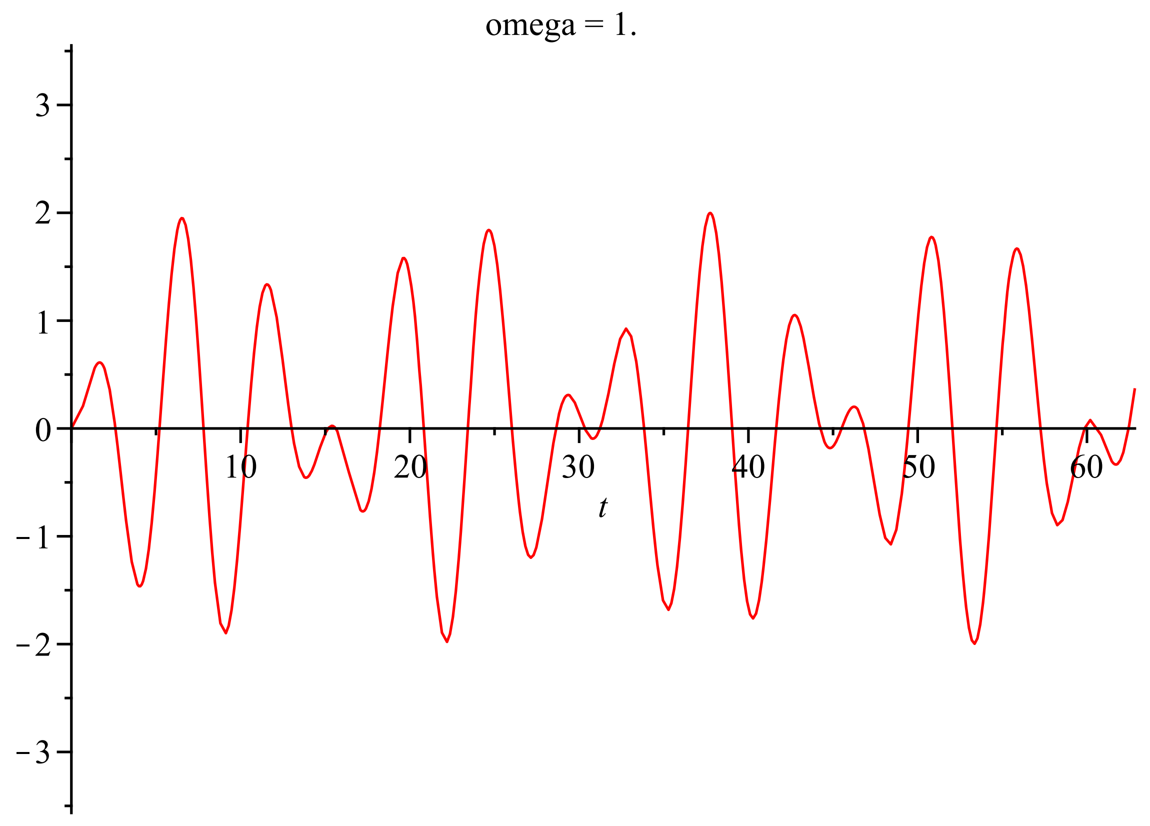

Figure 1: Equation (3) with $\omega = 1$

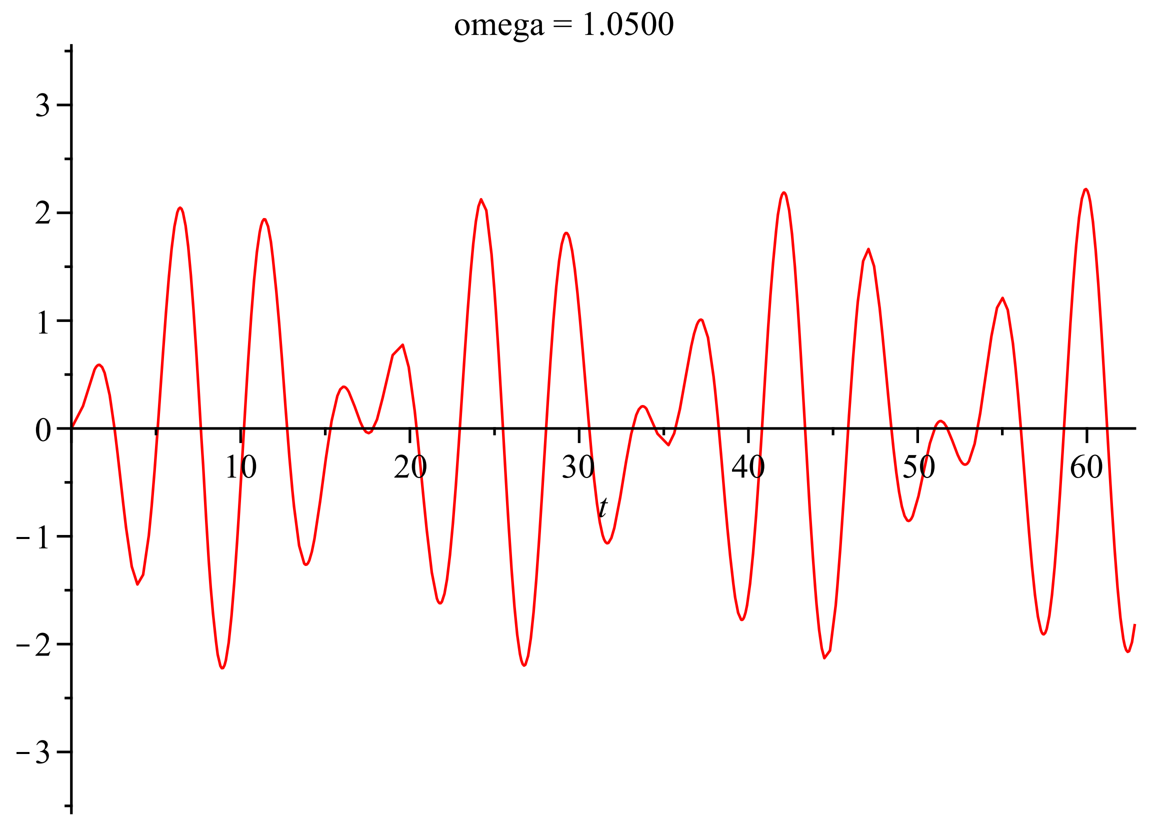

Figure 2: Equation (3) with $\omega = 1.05$

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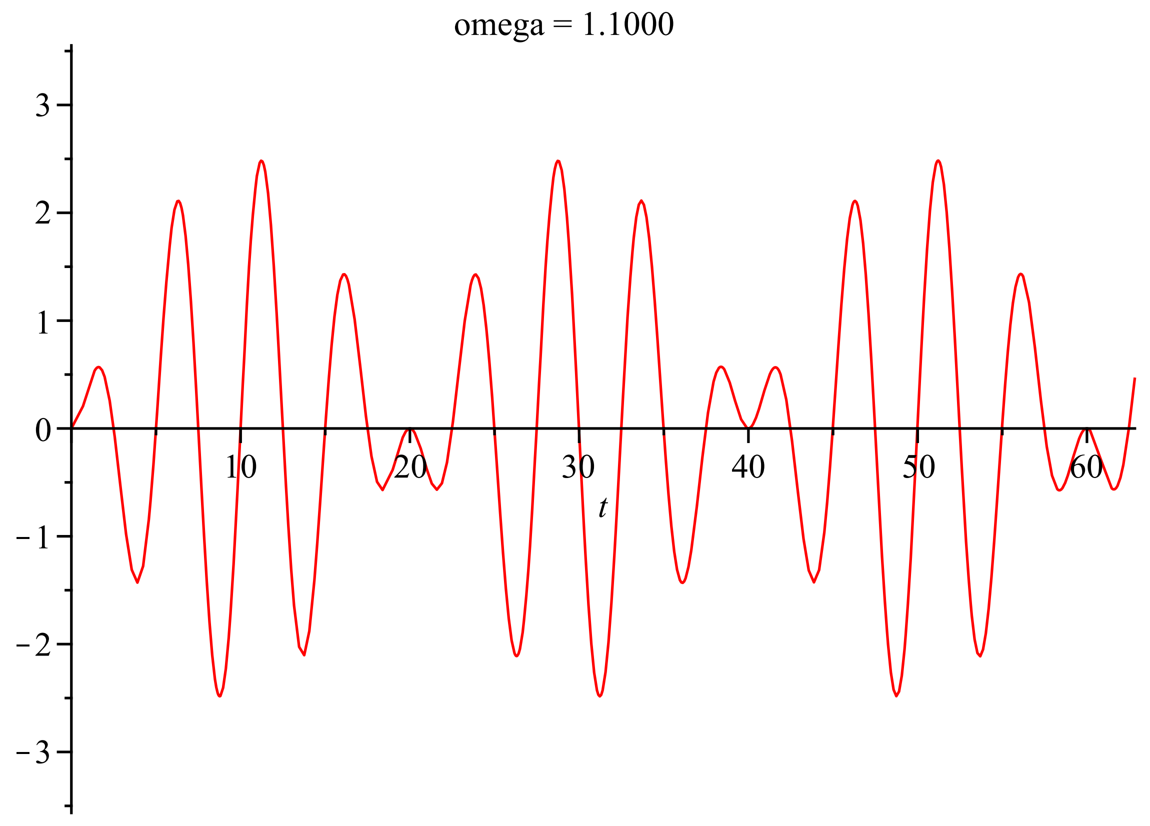

Figure 3: Equation (3) with $\omega = 1.1$

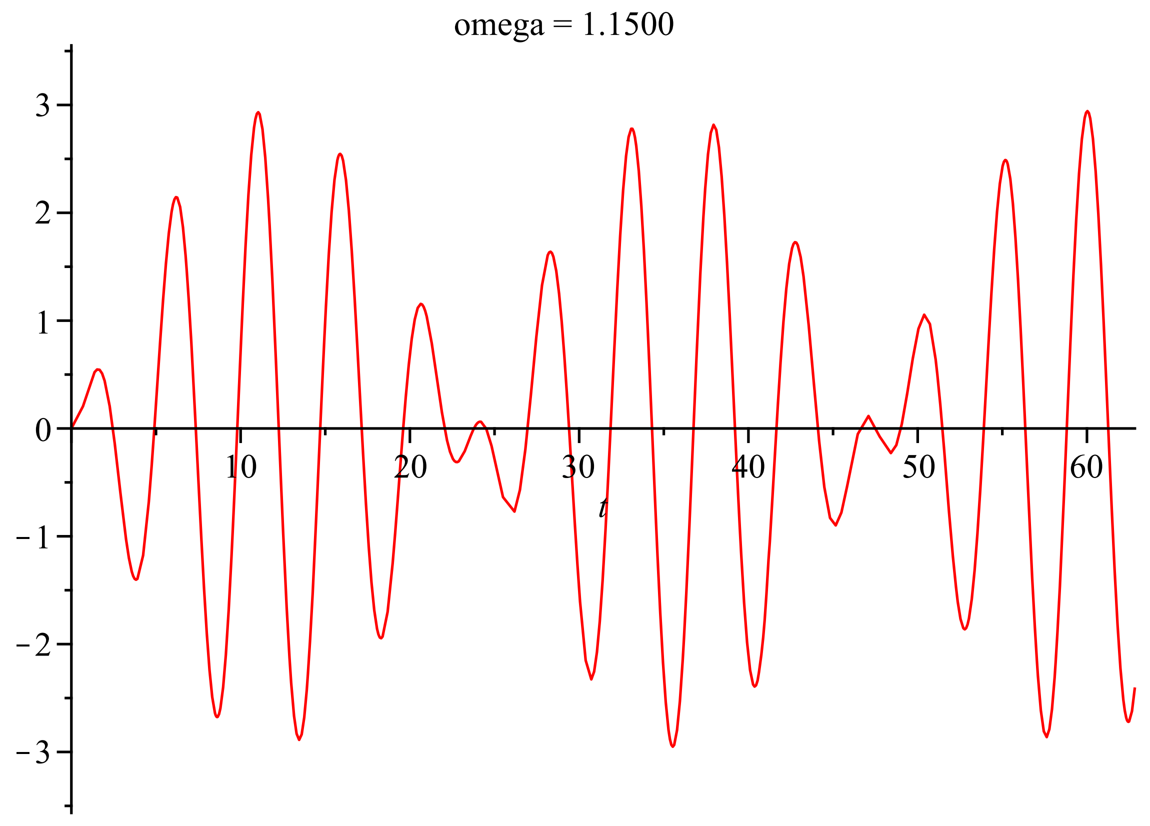

Figure 4: Equation (3) with $\omega = 1.15$

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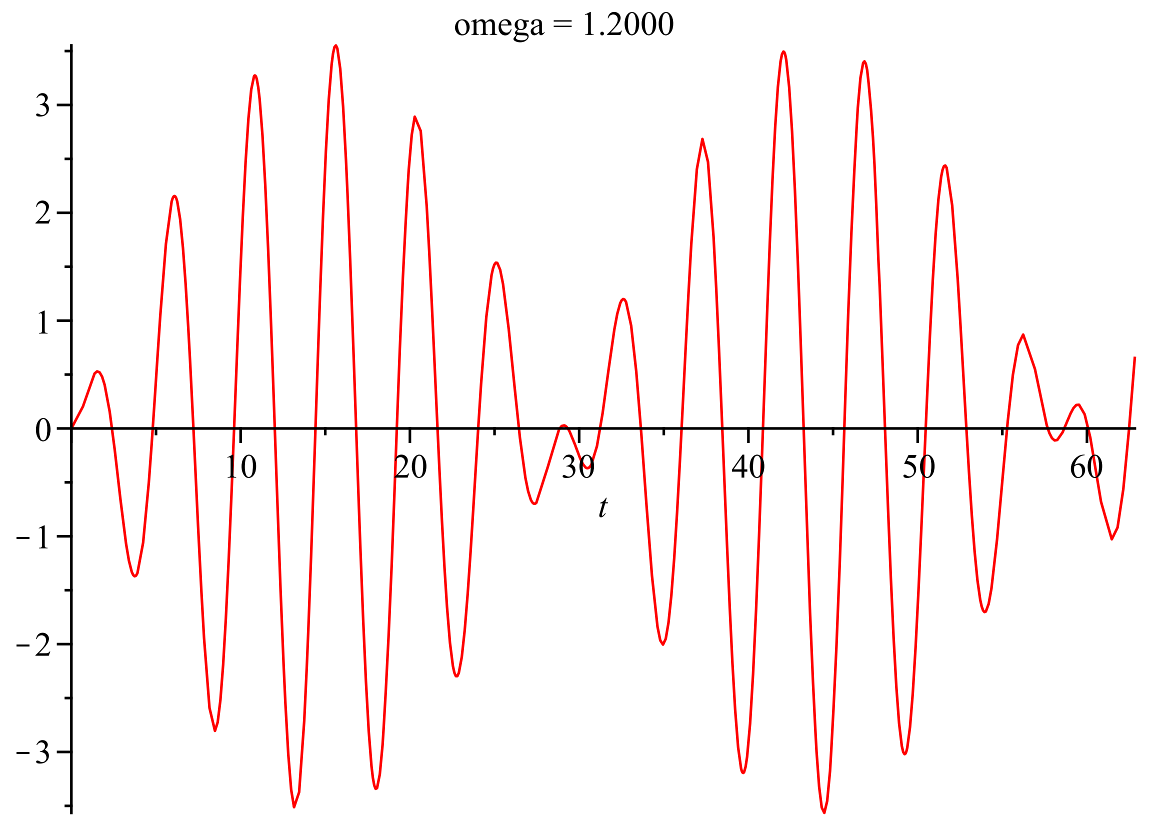

Figure 5: Equation (3) with $\omega = 1.2$

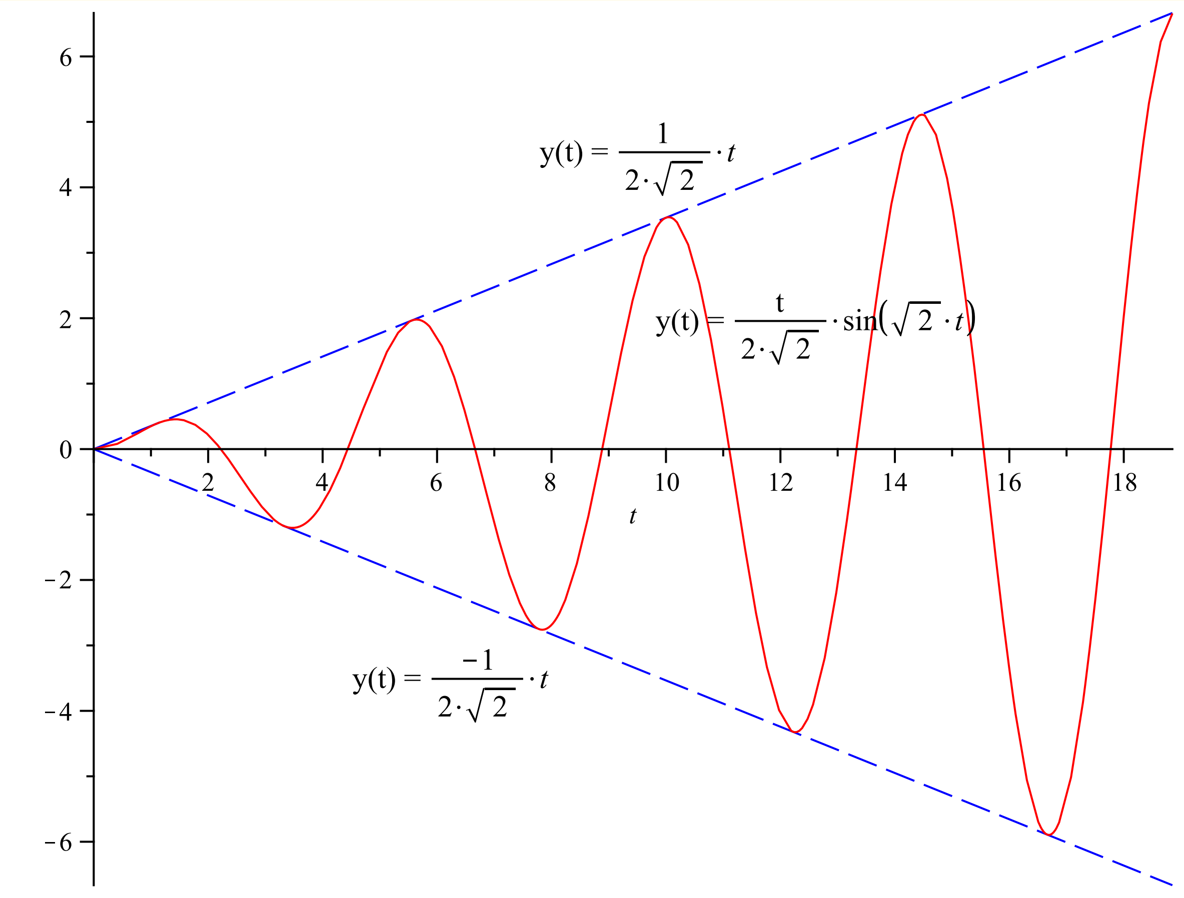

Figure 6: Solution of (2) with $\omega = \sqrt{2}$

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5. Homework

Section 6.3, page 593: 23, 25, 27, 28, 29, 31.