11-5 Networks

In the 1700s, the people of Königsberg, Germany (now Kaliningrad in Russia), used to enjoy walking over the bridges of the Pregel River. There were three landmasses and seven bridges over the river, as shown in Figure 11-50. These walks eventually led to the following problem.

Königsberg Bridge Problem

Is it possible to walk across all the seven bridges so that each bridge is crossed exactly once on the same walk?

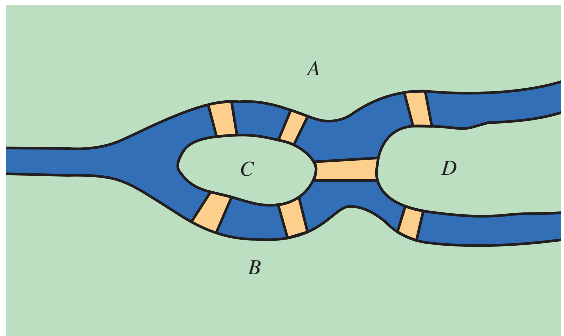


Figure 11-50

There is no restriction on where to start the walk or where to finish. Leonhard Euler became interested in this problem and solved it in 1736. He made the problem much simpler by representing the land masses and bridges in a **network**, as shown in the pink portion of Figure 11-51.

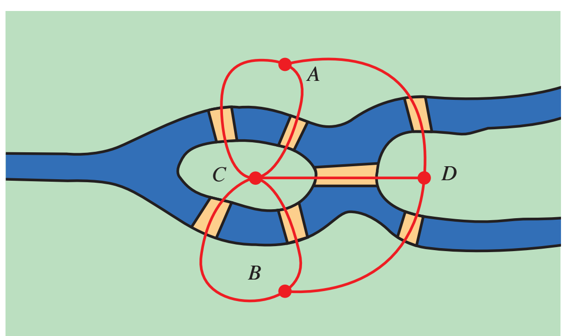


Figure 11-51

The points in a network are **vertices** and the curves are **arcs** also called **edges**. Using a network diagram, we can restate the Königsberg bridge problem as follows: *Is there a path through the network beginning at some vertex and ending at the same or another vertex such that each arc is traversed exactly once?* A network having such a path is **traversable**; that is, each arc is passed through exactly once. A tour on a traversable network in such a way that the starting point and the stopping point are the same is an **Euler circuit**.

We can walk around an ordinary city block, as illustrated in Figure 11-52(a), and because the starting point is the same as the stopping point, the network is an Euler circuit. We need not start at a particular point, and, in general, we can traverse any simple closed curve. Now consider walking around two city blocks and down the street that runs between them, as in Figure 11-52(b). To traverse this network, it is necessary to start at vertex *B* or *C*. Starting at points other than *B* or *C* might suggest that the figure is not traversable, but this is not the case, as shown in Figure 11-52(b). If we start at *B*, we end at *C*, and vice versa. Note that it is permissible to pass through a vertex more than once, but an arc may be traversed only once. Vertices *B* and *C* are endpoints of three arcs, and each of the other vertices are endpoints of two arcs.

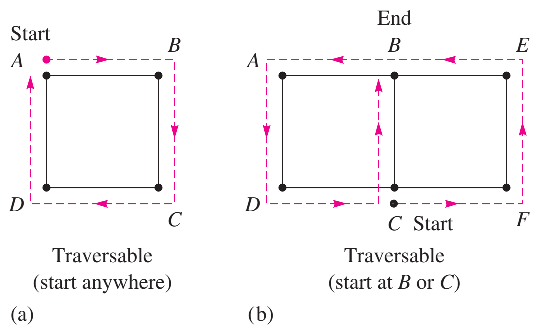


Figure 11-52

A traversable network is the type of network, or route, that a highway inspector would like to have if given the responsibility of checking out all the roads in a highway system. The inspector needs to traverse each road (arc) in the system but would save time by not having to make repeat journeys during an inspection tour. It would be feasible for the inspector to go through any town (vertex) more than once on the route. Consider the networks in Figure 11-53. Is it possible for the highway inspector to do the job with these networks without traversing a road twice?

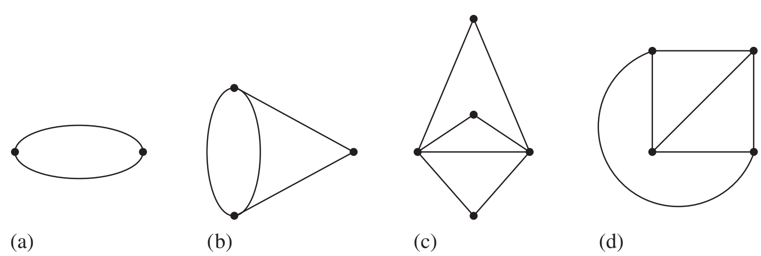


Figure 11-53

The first three networks in Figure 11-53 are traversable; the fourth network, (d), is not. Notice that the number of arcs *meeting at each vertex* in networks (a) and (c) is even. Any such vertex is an **even vertex**. If the number of arcs *meeting at a vertex* is odd, it is an **odd vertex**. In network (b), only the odd vertices will work as starting or stopping points. In network (d), which is not traversable, all the vertices are odd. If a network is traversable, each arrival at a vertex other than a starting or a stopping point requires a departure. Thus, *each vertex that is not a starting or stopping point must be even*. The starting and stopping vertices in a traversable network may be even or odd, as seen in Figure 11-53(a) and (b), respectively. Which networks, if any, form an Euler circuit?

Properties of a Network

In general, connected networks have the following properties.

1. If a network has all even vertices, it is traversable. Any vertex can be a starting point, and the same vertex must be the stopping point. Thus the network is an Euler circuit.
2. If a network has two odd vertices, it is traversable. One odd vertex must be the starting point, and the other odd vertex must be the stopping point.

NOW TRY THIS 11-13

a. Is there a traversable network with more than two odd vertices? Why or why not?
b. Is there a network with exactly one odd vertex?

Example 11-6

a. Which of the networks in Figure 11-54 are traversable?
b. Which of the networks in Figure 11-54 are Euler circuits?

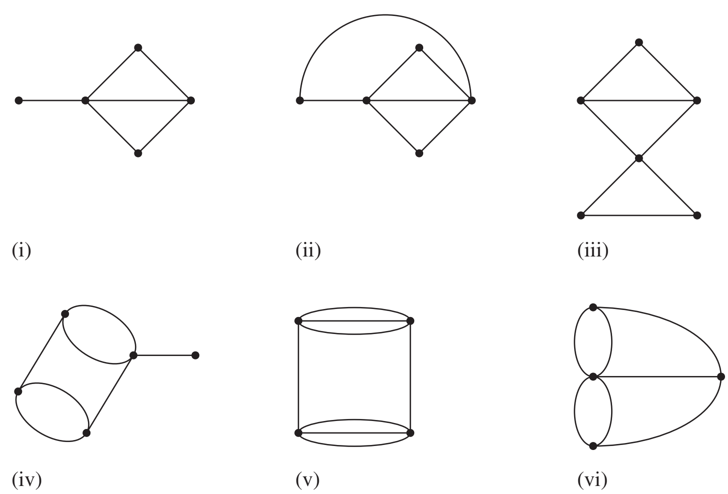


Figure 11-54

Solution **a.** Networks in (ii) and (v) all have even vertices and therefore are traversable. Networks in (i) and (iii) have exactly two odd vertices and are traversable. Networks in (iv) and (vi) have four odd vertices and are not traversable.
b. Networks (ii) and (v) are Euler circuits.

The network in Figure 11-54(vi) represents the Königsberg bridge problem. It has four odd vertices and consequently is not traversable. Hence, no walk configuration is possible to solve the problem.

A problem similar to the highway inspector problem involves a traveling salesperson. Such a person might have to travel networks comparable to those of the highway inspector. However, the salesperson is interested in visiting each town (vertex) only once, not necessarily in following each road. It is not known for which networks this can be accomplished. Can you find a route for the traveling salesperson for each network in Figure 11-54?

In many applications, *weighted networks* need to be considered. The student page gives an example of such networks. Answer question 29 on the student page.

A different type of application of network problems is discussed in Example 11-7.

Example 11-7

Look at the floor plan of the house shown in Figure 11-55. Is it possible for a security guard to go through all the rooms of the house and pass through each door exactly once?

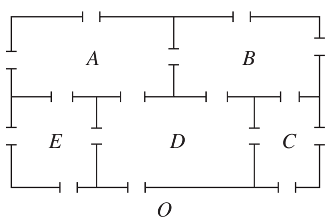


Figure 11-55

Solution Represent the floor plan as a network, as in Figure 11-56. Designate the rooms and the outside as vertices and the paths through the doors as arcs. The network has more than two odd vertices, namely, A, B, D, and O. Thus, the network is not traversable, and it is impossible to go through all the rooms and pass through each door exactly once.

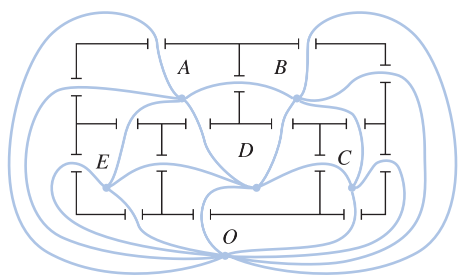
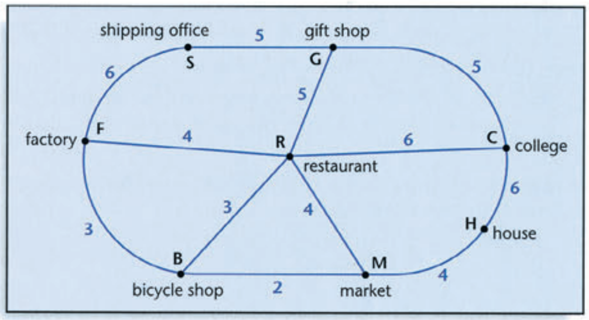


Figure 11-56

School Book Page WEIGHTED NETWORKS

▶ **A driver for a shipping company may deliver as many as 500 packages in a day, so drivers' routes must be efficiently planned. To route their trucks, a shipping company can use weighted networks like the one below, where the arcs are labeled with travel times in minutes between delivery points.**



26 How can you tell that the network above is not drawn to scale?

27 **Try This as a Class** Suppose a driver is at the shipping office and must deliver a package to the house. One route the driver could use is to go from S to G to C to H. This route can be written as S–G–C–H.

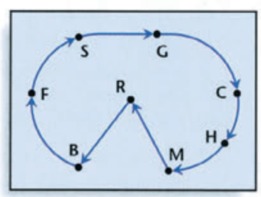
a. What is the travel time for this route?

b. What is another route the driver can use? What is the travel time for this route?

✓ QUESTION 28

... checks that you can interpret a weighted network.

28 **✓ CHECKPOINT** Suppose a driver must start at the shipping office, deliver packages to each delivery point, and return to the shipping office.



a. One route the driver can use is S–G–C–H–M–R–B–F–S. Find the travel time for this route.

b. What is another route the driver can use? What is the travel time for this route?

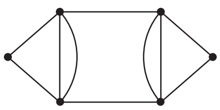
29 For the situation in Question 28, find the fastest route that starts at the shipping office, goes through each delivery point, and returns to the shipping office. What is its total travel time?

HOMEWORK EXERCISES ▶ See Exs. 20–25 on p. 448.

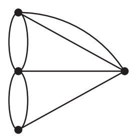
Source: Math Thematics, Book 2, McDougal Littell, 2005 (p. 444).

Assessment 11-5A

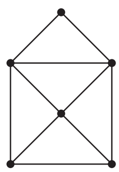
1. Which of the following networks are traversable? If the network is traversable, draw an appropriate path through it, labeling the starting and stopping vertices. Indicate which networks are Euler circuits.



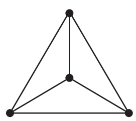
(a)



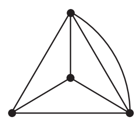
(b)



(c)

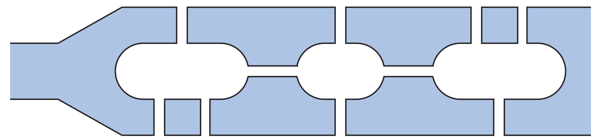


(d)

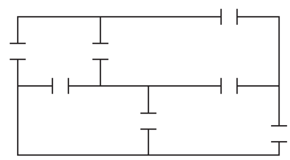


(e)

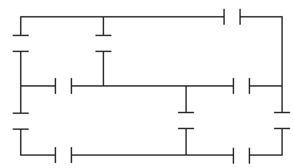
2. Which of the networks in exercise 1 can be traveled efficiently by a traveling salesperson, with no vertex visited more than once?
3. A city contains one river, three islands, and ten bridges, as shown in the following figure. Is it possible to take a walk around the city by starting at any land area, returning after visiting every part of the city, and crossing each bridge exactly once? If so, show such a path both on the original figure and on the corresponding network.



4. Refer to the following floor plans:



(i)



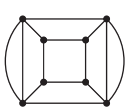
(ii)

   a. Draw a network that corresponds to each floor plan.
   b. Determine whether a person could pass through each room of each house by passing through each door exactly once. If it is possible, draw the path of such a trip.
5. Each network in exercise 1 separates the plane into several subsets. If R is the number of interior and exterior regions of the plane, V is the number of vertices, and A is the number of arcs, complete the following chart using each of the networks. (The first one is done for you.)

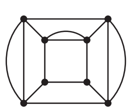
Network	R	V	A	$R + V - A$
(a)	6	6	10	2

Assessment 11-5B

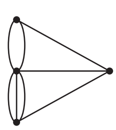
1. Which of the following networks are traversable? If the network is traversable, draw an appropriate path through it, labeling the starting and stopping vertices. Indicate which networks are Euler circuits.



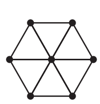
(a)



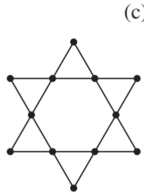
(b)



(c)

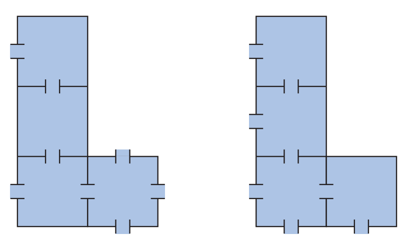


(d)



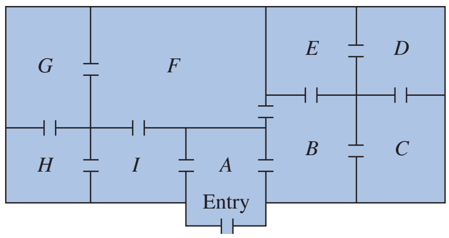
(e)

2. Which of the networks in exercise 1 can be traveled efficiently by a traveling salesperson, with no vertex visited more than once?
3. Refer to the following floor plans:



Can a person walk through each door once and only once and also go through both of the houses in a single path? If it is possible, draw the path of such a trip.

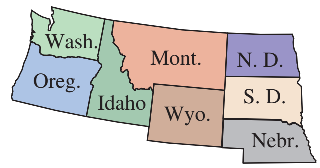
4. The following drawing represents the floor plan of an art museum. All tours begin and end at the entry. If possible, design a tour route that will allow a person to see every room but not go through any room twice.



5. Each network in problem 1 separates the plane into several subsets. If R is the number of interior and exterior regions of the plane, V is the number of vertices, and A is the number of arcs, complete the following chart using each of the networks. (The first one is done for you.)

Network	R	V	A	$R + V - A$
(a)	6	6	10	2

6. Molly is making her first trip to the United States and would like to tour the eight states pictured in the following figure. She would like to plan her trip so that she can cross each border between neighboring states exactly once—that is, the Washington–Oregon border, the Washington–Idaho border, and so on.

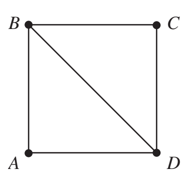


Is such a trip possible? If so, does it make any difference in which state she starts her trip?

Mathematical Connections 11-5

Communication

1. The following network is not an Euler circuit:



   a. Add two arcs to the network so that the resulting network will be an Euler circuit.
   b. Add exactly one arc so that the resulting network will be an Euler circuit.
   c. Explain a real-life application in which the answer in (b) is useful.
2. If you were commissioned to build an eighth bridge to make the Königsberg bridge network traversable, where would you build your bridge? Is there more than one location where you could build it? Explain why.

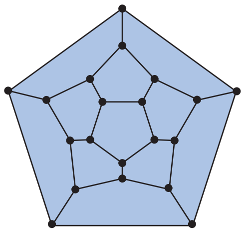
Open-Ended

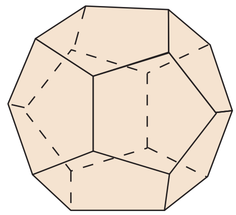
3. a. How would a postal worker's route differ from that of a traveling salesperson or a highway inspector?
   b. Consider a 10-square-block area of a city where a postal worker has to cover each side of each street. Design an efficient way to do this route.
4. Using as few vertices and arcs as possible, draw a network that is not traversable.
5. Draw a network that is not an Euler circuit and then add the least number of arcs (edges) possible so that the new network will be an Euler circuit.
6. One application of Euler circuits is the checking of parking meters. List other real-life applications that could involve the use of Euler circuits. In each case, give a concrete example and describe the corresponding Euler circuit.

Cooperative Learning

7. *Traveler's Dodecahedron* is a puzzle invented in 1857 by the Irish mathematician William Rowen Hamilton. It consists of a wooden dodecahedron (a polyhedron with 12 regular pentagons as faces) with a peg at each vertex of the dodecahedron. The 20 vertices are labeled with the names of different cities around the world. The solver of the puzzle is to find a path that starts at some city, travels along the edges, goes through each of the remaining cities exactly once, and returns to the starting city. The path traveled is to be marked by a string connecting the pegs.

a. Find a solution to the puzzle, first on the following network and then on the following dodecahedron. Compare your answer with those of other members of your group.



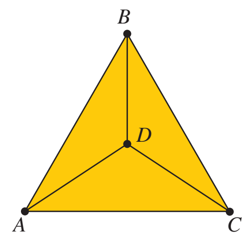


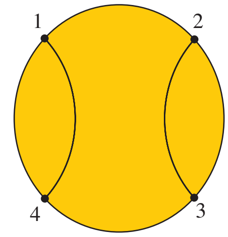
b. Play the following game with a partner. Draw a polyhedron that can be traversed in the way described earlier in this problem. Also draw a two-dimensional network for the polyhedron similar to the one shown in (a). Ask your partner to answer the question posed in (a) for the new polyhedron and the accompanying network. Then switch roles. The person who draws the polyhedron with the greatest number of vertices wins.

Questions from the Classroom

8. Nira drew a network modeling handshakes at a party. She represented each person by a vertex and a handshake between two people as an edge. She called the number of edges meeting at a vertex the degree of the vertex, and noticed that the sum of the degrees of all the vertices is always an even number. She would like to know if this is always true and why. How would you help her?

9. Dana thinks that the two networks shown are basically the same. Gila claims that the networks are quite different. Who is correct and why?





LABORATORY ACTIVITY

1. Take a strip of paper like the one in Figure 11-57(i). Give one end a half-twist and join the ends by taping them, as in Figure 11-57(ii). The surface obtained is a Möbius strip.

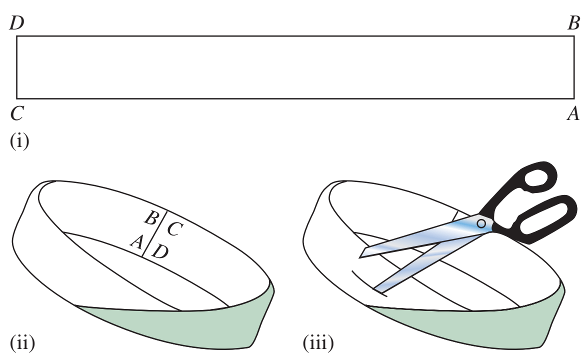


Figure 11-57

a. Use a pencil to shade one side of a Möbius strip. What do you discover?
b. Imagine cutting a Möbius strip all around midway between the edges, as in Figure 11-57(iii). What do you predict will happen? Now do the cutting. What is the result?
c. Imagine cutting a Möbius strip one-third of the way from an edge and parallel to the edge all the way through until you return to the starting point. Predict the result. Then do the cutting. Was your prediction correct?
d. Imagine cutting around a Möbius strip one-fourth of the way from an edge. Predict the result. Then do the cutting. How does the result compare with the result of the experiment in (c)?

2. a. Take a strip of paper and give it two half-twists (one full twist). Then join the ends together. Answer the questions in part 1.
   b. Repeat the experiment in (a) using three half-twists.
   c. Repeat the experiment in (a) using four half-twists. What do you find for odd-numbered twists? Even-numbered twists?
3. Take two strips of paper and tape each of them in a circular shape. Join them as shown in Figure 11-58.

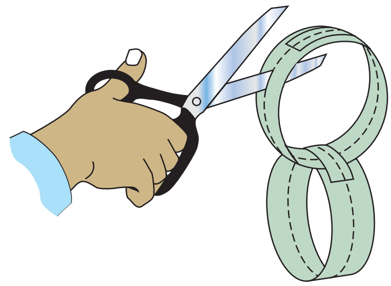


Figure 11-58

   a. What happens if you cut completely around the middle of each strip as shown?
   b. Repeat part (a) if both strips are Möbius strips. Does it make any difference if the half-twists are in opposite directions?
   c. Repeat part (a) if one strip is a Möbius strip and the other is not. What happens when you cut the strips?

BRAIN TEASER Given three buildings *A*, *B*, and *C*, as shown in Figure 11-59, and three utility centers for electricity (*E*), gas (*G*), and water (*W*), determine whether it is possible to connect each of the three buildings to each of the three utility centers without crossing lines.

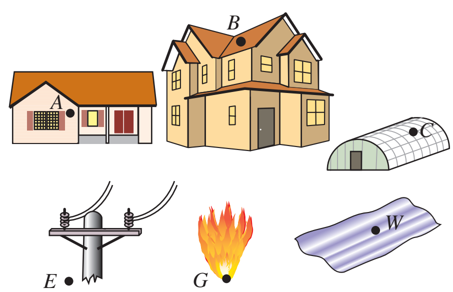


Figure 11-59