International Journal of Intelligent Technology and Applied Statistics
Vol.2, No.2 (2009) pp.57-64

A Note on the Maximum Likelihood Estimation for the Generalized Gamma Distribution Parameters under Progressive Type-II Censoring

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ABSTRACT

Based on the progressively type-II censored sample, the maximum likelihood estimates for the parameters of the generalized Gamma distribution are studied. In dealing with the parameter confounding in this distribution, a new re-parametrization is proposed to reformulate the generalized Gamma distribution to sustain the numerical stability in the search algorithm for the maximum likelihood estimation. Intensive simulation is conducted to evaluate the maximum likelihood estimates in terms of the mean squared error and bias via the proposed reparameterization. Finally, a real data set from censored times to breakdown of an insulating fluid is used to demonstrate its applicability.

Keywords: Maximum likelihood estimate; Parameter confounding; Probability integral transform; Type-II censoring

1. Introduction

The three-parameter generalized gamma (GG) distribution has probability density function and distribution function respectively defined as:

$$f(t,\theta) = (\beta/\Gamma(k))\lambda^{\beta k} t^{\beta k-1} e^{-(\lambda t)^{\beta}}, t > 0 \tag{1}$$

and

$$F(t,\theta) = \int_0^{(\lambda t)^{\beta}} \frac{1}{\Gamma(k)} w^{k-1} exp(-w) dw = \frac{IG((\lambda t)^{\beta}, k)}{\Gamma(k)}, t > 0, \tag{2}$$

where $\theta = (\beta, \lambda, k)$, $\beta > 0$ and $k > 0$ are the shape parameter, $\lambda > 0$ is the scale parameter and $IG(x,k) = \int_0^x t^{k-1} e^{-t} dt$.

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When $k = 1$, the GG distribution defined above reduced to the Weibull distribution and when $\beta = 1$, the GG distribution is the gamma distribution. The GG distribution was introduced by Stacy [9] and has been considered as a useful life distribution model since then. For complete random sample, Stacy and Mihram [10] studied some basic properties of the density. Bain and Weeks [4] developed one-sided tolerance limit and confidence limit for each of these three parameters, respectively, assuming that only the corresponding parameter is unknown. Parr and Webster [8] provided tests for the Weibull and (or) the exponential model against alternative model as GG distribution model. Hager and Bain [6] extended the study the properties of maximum likelihood estimators of the GG distribution model and provided discriminating process between the Weibull model and the GG model.

In industrial life testing and medical survival analysis, progressively censored schemes get special attention in the last two decades. In the progressively censored schemes, the tested items are allowed to be withdrawn at some other times before the termination time of life test. The allowance lets experimenters have a chance to observe more extreme lifetimes compared with the typically censored schemes. Moreover, some test facilities can be freed up from the life test for other uses. See Balakrishnan and Aggarwala [2] for more information about progressively censoring schemes and the applications. Some referred papers regarding the reliability applications with progressively censoring schemes can be found in Cohen [5], Thomas and Wilson [11], Tse and Yuen [14], Tse *et al.* [12], Wu and Chang [16], Abd-Elfattah and Assar [1] and Tsai *et al.* [13].

The primary purpose of this paper is to study the performance of the parameter estimation under the progressive type-II censoring by the maximum likelihood method. The performance evaluation of the parameter estimation is implemented numerically in terms of mean squared error (MSE) and bias. The rest of this paper is organized as follows. The progressive type-II censoring scheme and maximum likelihood estimates for the parameters are discussed in Section 2. In Section 3, an intensive simulation study is conducted to illustrate this estimation. In Section 4, an example is used for illustration followed by the conclusion remarks in Section 5 .

2. Progressive type-II censoring and parameter estimation

2.1 Progressive type-II censoring sample

Let n items be placed on a life test simultaneously at time $t_0 = 0$ and the test terminates when the m^{th} failure unit is observed, where $1 \leq m \leq n$. At the i^{th} failure time t_i, r_i of surviving units are randomly removed, where $i = 1, 2, ..., m$. Hence, at the end time of life testing, the remaining $r_m = n - \sum_{i=1}^{m-1} r_i - m$ surviving units are all removed. Therefore, a progressively type II censored sample can be denoted as $\{t_i, r_i\}, i = 1, 2, ..., m$. If $r_i = 0, i = 1, 2, ..., m-1$, and $r_m > 0$, then the progressively type II censored sample reduces to a conventional type II censored sample of $t_1, t_2, \cdots, t_m$ and $m < n$.

Given a progressively type II censored sample, $\{t_i, r_i\}, i = 1, 2, ..., m$ of size n the likelihood function can be constructed as the following:

$$L(\theta) \propto \prod_{i=1}^{m} f(t_i, \theta) \left[1 - F(t_i, \theta)\right]^{r_i}. \tag{3}$$

2.2 Maximum likelihood estimation

Assuming that a progressive type II censored sample, $\{t_i, r_i\}, i = 1, 2, ..., m$ of size n, as described in Section 2, from the GG distribution defined in model (2), the likelihood function in (3) can be specified as follows:

$$L(\beta, \lambda, k) \propto \left[\frac{\beta\lambda^{\beta k}}{\Gamma(k)}\right]^m \prod_{i=1}^{m} t_i^{\beta k-1} exp(-(\lambda t_i)^\beta) \left[1 - \frac{IG((\lambda t)^\beta, k)}{\Gamma(k)}\right]^{r_i}. \tag{4}$$

By setting the derivatives of the log-likelihood function with respective to β, λ or k to zero, the maximum likelihood estimates (MLEs) of β, λ or k are the solutions to the following likelihood equations

$$\frac{m}{\beta} = \sum_{i=1}^{m}[(\lambda t_i)^\beta ln(\lambda t_i) - k ln(t_i\lambda) + \frac{r_i ln(\lambda t_i)(t_i\lambda)^{\beta k} exp(-(t_i\lambda)^\beta)}{\Gamma(k) - IG((t_i\lambda)^\beta, k)}], \tag{5}$$

$$mk = \sum_{i=1}^{m}[\lambda^\beta t_i^\beta + \frac{r_i exp(-(t_i\lambda)^\beta)(\lambda t_i)^{\beta k}}{\Gamma(k) - IG((t_i\lambda)^\beta, k)}], \tag{6}$$

and

$$\frac{n\Gamma'(k)}{\Gamma(k)} = m\beta ln(\lambda) + \sum_{i=1}^{m}[\beta ln(t_i)] + \sum_{i=1}^{m} r_i[\frac{\Gamma'(k) - \Psi((t_i\lambda)^\beta, k)}{\Gamma(k) - IG((t_i\lambda)^\beta, k)}], \tag{7}$$

where $\Psi((t_i\lambda)^\beta, k) = \int_0^{(t_i\lambda)^\beta} w^{k-1} ln(w) e^{-w} dw$. These equations reduced to likelihood equations given by Hager and Bain [6] when $r_i = 0, i = 1, 2, 3, ..., m$. There is no closed form of solution to the above equations, and iterative numerical search can be used to obtain the MLEs from the above likelihood equations.

2.3 GG reparameterization

Applying the likelihood estimation described in Subsection 2.2 for the simulation study in Section 3 and the real data application in Section 4, we find that the maximum likelihood estimation (MLE) is very unstable primarily because of the parameter confounding in the GG distribution. This exercise confirmed with the study of the MLE by Hager and Bain [6] for complete random sample. We experiment extensively to reformulate the GG distribution paramters in many different ways and find out that the reformulation of

$$\theta_{new} = (\alpha, \lambda, k) \tag{8}$$

where $\alpha = \beta \times k$ produced the most satisfactorily numerical stability. Therefore, the likelihood function (4) is reparameterized in terms of θ_{new} for the simulation study in the Section 3 and also for fitting the real data set given in the Section 4.

3. Simulation study

The progressive type-II censored samples where generated by the algorithm developed in Balakrishnan and Sandhu [3] to generate the progressive type-II censored sample from uniform [0,1] then followed by the probability integral transformation to generate progressive type-II censored sample for the GG distribution. According to Section 2.3, the GG distribution is reparameterized in terms of θ_{new} given in (8). For simplicity, we consider the simulation parallel to the real data present in Section 4. The GG distribution parameters for the simulation are given as $\theta_{new} = (\alpha, \lambda, k) = (0.843, 0.068, 0.034)$. The input parameters for the simulation study are obtained via the MLEs of θ_{new} for the GG distribution based on the data in Table 2.

In order to confirm the data generation, we track the generated censoring time t with the observed time in Table 2. The plot in the upper-left corner in Figure 1 illustrates the outcome from a simulation. In this plot, the dots are the simulated time v.s. the observed time, the solid line is the 1-1 line and the broken line is the superimposed regression line of

$$t_{smulated} = \beta_0 + \beta_1 t_{observed}. \tag{9}$$

The plot confirms the validity of this data simulation.

With the simulated censored sample, we keep tracking this data generation as well as the maximum likelihood estimation applied to estimate the parameter vector θ. Ten thousand simulations are conducted, the rest of the plots in Figure 1 summarize the distributions of these ten thousand estimates for parameters α, λ and k, respectively. The figures labeled as β_0 and β_1 are the distributions for the estimates of the constant and the slope in the regression model of (9) for the data generation. They show that the distributions for β_0 and β_1 are approximately symmetrical about mean 0 and 1, respectively. And the parameter estimates for $\hat{\theta} = (\hat{\alpha}, \hat{\lambda}, \hat{k})$ are generally biased.

To numerically illustrate this simulation, Table 1 summarizes the simulation results. In this table, the bias is the average of the 10000 parameter MLEs subtracted by the corresponding input parameter for simulation. Squared relative error(SRE)

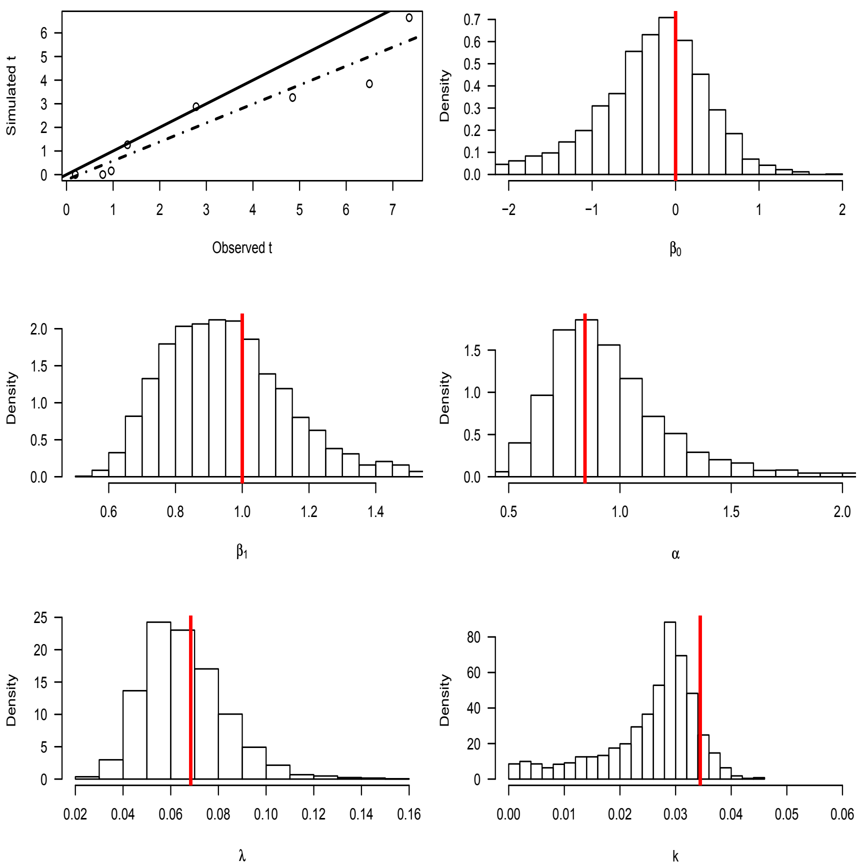

Figure 1. Histograms for the simulated parameters. Parameters of α,λ and k are for the GG distribution. β_0 and β_1 are for the simulation validation regression.

Table 1. Bias, squared relative error(SRE) and MSE from the 10000 simulations. Note that the relative error for β_0 is not defined and denoted by NA. The results have been scaled by 100.

	α	λ	k	β_0	β_1
Bias	11.39	-0.25	-0.89	-33.78	-4.27
SRE	14.10	6.41	13.09	NA	4.03
MSE	10.02	0.03	0.02	11.44	15.26

Table 2. Progressively censored sample generated from the Log-times to Breakdown data on insulating fluid tested at 34 kilovolts by Nelson [6] (table 6.1).

i	1	2	3	4	5	6	7	8
$x_{i,n}$	-1.6608	-.2485	-.0409	.2700	1.0224	1.5789	1.8718	1.9947
R_i	0	0	3	0	3	0	0	5

is calculated as the average of the 10000 squared ratio of the difference between parameter MLE and corresponding input parameter over the corresponding input parameter, i.e.

$$SRE = mean\left[(\frac{\hat{\eta}-\eta}{\eta})^2\right], \tag{10}$$

where η presents any component in θ_{new} and the MSE is the mean square error for the estimator. Table 1 contains the simulation results which indicates the simulation performs well after reparameterization.

4. Real data analysis

Original experiments and data were from Nelson [7](p.228, Table 6.1) for the times to breakdown of an insulating fluid in an accelerated test conducted at various test voltages. In analyzing these data, Nelson considered a Weibull model so that the corresponding log-times to breakdown could be treated as extreme-value observations.

To illustrate the type-II progressively censored sample, Viveros and Balakrishnan [15] randomly generated a type-II progressively censored sample of size $m = 8$ from the $n = 19$ observations recorded at 34 kilovolts in Nelson's table 6.1. The observations and the three-stage removal pattern applied are reproduced in Table 2. As seen from this table, 11 failure times are censored and 8 are observed in this sample.

We use the reformulation for the parameters of the GG distribution discussed in Section 2.3 as $\theta_{new} = (\alpha, \lambda, k)$, where $\alpha = \beta \times k$. The maximum likelihood estimate of θ_{new} is found as $\hat{\theta}_{new} = (\hat{\alpha}, \hat{\lambda}, \hat{k}) = (0.843, 0.068, 0.034)$. Therefore, the maximum likelihood estimate for β is $\hat{\beta} = \frac{\hat{\alpha}}{\hat{k}} = 24.479$. The approximated standard errors for the maximum likelihood estimator of θ_{new} are (0.269,0.437,10.311) which then produces the t-statistics for $\hat{\theta}_{new}$ as (3.139,0.156,0.003).

5. Conclusions

In this paper, the maximum likelihood estimations of the generalized Gamma distribution parameters have been developed under progressive type-II censoring. We find out that the original parameters $\theta = (\beta, \lambda, k)$ in the GG distribution are confounded and the numerical search for the maximum likelihood estimate is very unstable. Many different attempted re-parametrization processes have been studied and the new reformation of parameters $\theta_{new} = (\alpha, \lambda, k)$ with $\alpha = \beta \times k$ provides the most numerical stability for searching the maximum likelihood estimations of the GG distribution parameters.

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